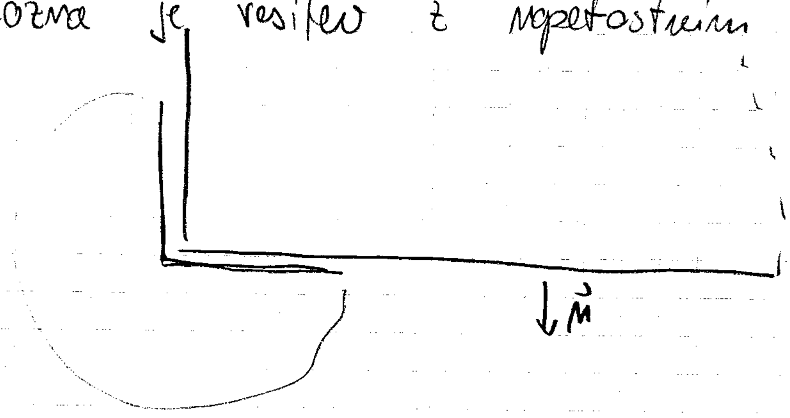
1. Poprovni kolokvij

1) Možna je rešitev z napetostnim tenzorjem:



- Računamo silo na osmino prostora. Najprej prispevek ravnine x-y:

$$\vec{E}(r) = \frac{e \cdot r}{4\pi\varepsilon_0 R^3} \cdot \hat{e}_r \qquad r<R$$

$$\vec{E}(r) = \frac{e}{4\pi\varepsilon_0 r^2}\, \hat{e}_r \qquad r>R$$

$$\vec{F}_1 = \int \overset{\leftrightarrow}{T} \cdot \vec{n}\, dS$$

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} \vec{E}^2 \right) \qquad \vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$T_{ij} \cdot n_j = -\varepsilon_0 \left(E_i \underset{\substack{\parallel \\ 0 \text{ na ravnini } x\text{-}y}}{E_z} - \frac{1}{2}\delta_{iz} E^2 \right) = \varepsilon_0 \frac{1}{2} \delta_{iz} E^2$$

$$\vec{F}_1 = \int_0^{\pi/2} d\varphi \int_0^{\infty} dr \cdot r \cdot \varepsilon_0 \cdot \frac{1}{2} E^2(r) \cdot \hat{e}_z = \frac{\pi}{4}\varepsilon_0 \cdot \left(\int_0^R \frac{e^2 r^2}{(4\pi\varepsilon_0)^2 R^6}\, r dr + \right.$$

$$\left. + \int_R^{\infty} \frac{e^2}{(4\pi\varepsilon_0)^2 r^4} \cdot r dr \right) \hat{e}_z = \frac{\pi}{4} \frac{\varepsilon_0 \cdot e^2}{4^2\pi^2\varepsilon_0^2} \left(\frac{R^4}{4R^6} + \frac{1}{2R^2} \right) e_z =$$

$$= \frac{3e^2}{256\pi\varepsilon_0 R^2}\,\hat{e}_z$$

Analogno velja za ostali ploskvi, zato:

$$\vec{F}_{sk.} = \frac{3e^2}{256\pi\varepsilon_0 R^2}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

• Lahko pa silo izračunamo tudi direktno:

$$\vec{F} = \int dV\,\vec{E}(\vec{x})\cdot\rho(\vec{x}) = \int_0^{\pi/2} d\vartheta \int_0^{\pi/2} d\varphi \int_0^R dr\cdot r^2 \sin\vartheta \cdot$$

$$\cdot\ \frac{e}{4\pi\varepsilon_0 R^3}\begin{pmatrix} r\cdot\sin\vartheta\cos\varphi \\ r\cdot\sin\vartheta\sin\varphi \\ r\cdot\cos\vartheta\end{pmatrix}\frac{3e}{4\pi R^3}$$

$$= \frac{3e^2}{4^2\pi^2\varepsilon_0 R^6}\cdot\frac{R^4}{4}\cdot\begin{pmatrix}\pi/4\\ \pi/4\\ \pi/2\cdot\frac{1}{2}\end{pmatrix} = \frac{3e^2}{256\pi\varepsilon_0 R^2}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

2) $\vec{E} = \frac{e}{4\pi\varepsilon_0 r^2} \cdot \hat{e}_r$

$\vec{B} = B \cdot \hat{e}_z$

$\vec{g} = \frac{1}{c^2} \cdot \vec{S} = \varepsilon_0 \cdot \vec{E} \times \vec{B} = \varepsilon_0 \frac{e}{4\pi\varepsilon_0 r^2} \cdot B \cdot \underbrace{\hat{e}_r \times \hat{e}_z}_{-\sin\vartheta \cdot \hat{e}_\varphi}$

$\vec{\gamma} = \vec{r} \times \vec{g} = \frac{eB}{4\pi r^2} \cdot r \cdot (\hat{e}_r \times \hat{e}_\varphi) \cdot (-\sin\vartheta) =$

$= \frac{eB}{4\pi r} \sin\vartheta \cdot \hat{e}_\vartheta$

$\vec{\Gamma} = \int dV \vec{\gamma}$, zaradi simetrije mora biti $\vec{\Gamma} = \Gamma_z \cdot \hat{e}_z$

$\Gamma_z = \int_{R_1}^{R_2} dr \cdot 2\pi r^2 \cdot \int_0^\pi \sin\vartheta \, d\vartheta \cdot \frac{eB}{4\pi r} \sin\vartheta \cdot$ $(-1) \sin\vartheta$

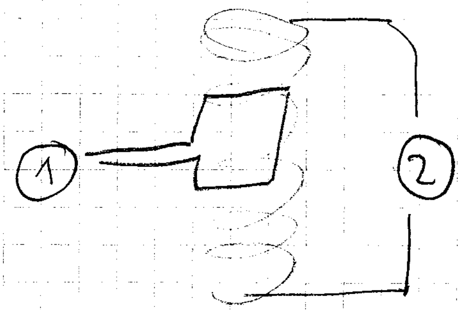
$= -\frac{eB}{2} \left(\frac{R_2^2}{2} - \frac{R_1^2}{2} \right) \underbrace{\int_0^\pi \sin^3\vartheta \, d\vartheta}_{\int_{-1}^{1} (1-\cos^2\vartheta) \, d\cos\vartheta = \frac{4}{3}}$

$= -\frac{eB}{3} (R_2^2 - R_1^2)$

$w = \frac{\Gamma_z}{J} = -\frac{eB(R_2^2 - R_1^2)}{3J}$

3) Medsebojna induktivnost je simetrična:

$L_{12} = L_{21}$



$$\Phi_1 = a^2 \cdot B_2 = a^2 \cdot \frac{\mu_0 I_2 \cdot N}{l}$$

$$\Phi_1 = L_{12} \cdot I_2$$

$$\Downarrow$$

$$\underline{L_{12} = \mu_0 \cdot N \cdot a^2 / l}$$

a) $U_2 = -L_{21} \cdot \dot{I}_1 = -\mu_0 N a^2/l \, \omega I_0 (-\sin \omega t) =$

$= \left(\mu_0 \cdot N a^2 \cdot \omega I_0 / l \right) \cdot \sin \omega t$

b) $U_2 = -L_{21} \cdot \dot{I}_1 - L_{22} \cdot \dot{I}_2$

L_{22}: lastna induktivnost tuljave,

$$L_{22} = \frac{\mu_0 \cdot N^2 \cdot \pi R^2}{l}$$

$U_2 = 0 \Rightarrow I_2 = -I_1 \cdot \frac{L_{21}}{L_{22}} = -I_1 \cdot \frac{\cancel{\mu_0} N a^2/\cancel{l}}{\cancel{\mu_0} N^2 \cdot \pi R^2/\cancel{l}}$

$$= \underline{-I_1 \cdot \frac{a^2}{N \cdot \pi R^2}}$$

c) $$U_1 = -\underbrace{L_{11}}_{\text{Ne poznamo}} \cdot \dot{I}_1 - L_{12} \cdot \dot{I}_2 =$$

$$= -L_{11} \cdot \dot{I}_1 - L_{12} \cdot \left(-\dot{I}_1 \frac{L_{21}}{L_{22}} \right) =$$

$$= \underbrace{-L_{11} \cdot \dot{I}_1}_{\text{Začetna napetost}} + \underbrace{\left(\frac{L_{12}^2}{L_{22}} \right) \cdot \dot{I}_1}_{\text{Sprememba po sklenitvi tuljave}}$$

Nasproten predznak $\Rightarrow$ Amplituda se zmanjša.

$$\Delta U^0 = -\frac{L_{12}^2}{L_{22}} \cdot \omega \cdot I^0 = -\frac{\mu_0^2 N^2 a^4 / l^2}{\mu_0 N^2 \cdot \pi R^2 / l} \cdot \omega I_0 =$$

$$= -\frac{\mu_0 a^4}{\pi R^2 \cdot l} \cdot \omega I_0$$

2. popravni kolokvij

1) Valovanji seštejmo z različnima amplitudama:

$$\vec{E} = \vec{E}^{(1)} e^{i(kx + \varkappa z - \omega t)} + \vec{E}^{(2)} e^{i(kx - \varkappa z - \omega t)}$$

Veljati mora

$$\vec{E}^{(1)} \perp \begin{pmatrix} k \\ 0 \\ \varkappa \end{pmatrix} \quad \text{in } \vec{E}^{(2)} \perp \begin{pmatrix} k \\ 0 \\ -\varkappa \end{pmatrix} :$$

$$\vec{E}^{(1)} = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + B \begin{pmatrix} -\varkappa \\ 0 \\ k \end{pmatrix}$$

$$\vec{E}^{(2)} = C \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + D \begin{pmatrix} \varkappa \\ 0 \\ k \end{pmatrix}$$

Robni pogoj: $E_x = E_y = 0$ pri $\begin{cases} z = 0 \\ z = d \end{cases}$ in

$$E_x(z=0) = (-\varkappa \cdot B + \varkappa \cdot D) e^{i(kx - \omega t)}$$

$$\Rightarrow B = D$$

$$E_y(z=0) = (A + C) e^{i(kx - \omega t)}$$

$$\Rightarrow A = -C$$

$$\vec{E} = A \cdot \hat{e}_y \cdot \left(e^{i(kx + \varkappa z - \omega t)} - e^{i(kx - \varkappa z - \omega t)}\right)$$

$$+ B \cdot \begin{pmatrix} -\varkappa \left(e^{i(kx + \varkappa z - \omega t)} - e^{i(kx - \varkappa z - \omega t)}\right) \\ 0 \\ k \left(e^{i(kx + \varkappa z - \omega t)} + e^{i(kx + \varkappa z - \omega t)}\right) \end{pmatrix}$$

$$= 2i \cdot A \cdot \hat{e}_y \cdot \sin(\varkappa z) \cdot e^{i(kx-\omega t)}$$

$$+ B \cdot \begin{pmatrix} -\varkappa \cdot 2i \cdot \sin(\varkappa z) \\ 0 \\ k \cdot 2 \cdot \cos(\varkappa z) \end{pmatrix} e^{i(kx-\omega t)}$$

Robni pogoj pri $\underline{z=d}$: $E_x = E_y = 0$

$$\Rightarrow \sin(\varkappa d) = 0$$

$$\underline{\underline{\varkappa = n \cdot \frac{\pi}{d}}}$$

<u>Rešitve:</u>

(i) $\vec{E}_{(i)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin\left(n \frac{\pi}{d} z\right) e^{i(kx-\omega t)}$

$n = 1, 2, \ldots$ <u>TE-rešitev</u>

(ii) $\vec{E}_{(ii)} = \begin{pmatrix} -i \frac{n\pi z}{d} \cdot \sin\left(\frac{n\pi z}{d}\right) \\ 0 \\ k \cdot \cos\left(\frac{n\pi z}{d}\right) \end{pmatrix} \cdot e^{i(kx-\omega t)}$

<u>$n = 1, 2, \ldots$ TM-rešitev</u>

<u>$n = 0$: TEM-rešitev</u>

Frekvenca:

$$\omega = c \cdot |\vec{k}| = c\sqrt{k^2 + \left(\frac{n \cdot \pi}{d}\right)^2}$$

Magnetno polje:

$$\vec{B}^0 = \frac{1}{\omega}\,\vec{k} \times \vec{E}^0$$

Rešitev (i): TE

$$\vec{B} = \frac{A}{\omega}\begin{pmatrix} k \\ 0 \\ \varkappa \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(kx + \varkappa z - \omega t)} -$$

$$- \frac{A}{\omega}\begin{pmatrix} k \\ 0 \\ -\varkappa \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(kx - \varkappa z - \omega t)}$$

$$= \frac{A}{\omega}\begin{pmatrix} -\varkappa \cdot e^{i\varkappa z} - \varkappa \cdot e^{-i\varkappa z} \\ 0 \\ k \cdot e^{i\varkappa z} - k \cdot e^{-i\varkappa z} \end{pmatrix} e^{i(kx - \omega t)}$$

$$= \frac{2iA}{\omega}\begin{pmatrix} i \cdot \varkappa \cdot \cos(\varkappa z) \\ 0 \\ k \cdot \sin(\varkappa z) \end{pmatrix} e^{i(kx - \omega t)}$$

Rešitev (ii): TM

$$\vec{B} = \frac{B}{\omega}\begin{pmatrix} k \\ 0 \\ \varkappa \end{pmatrix} \times \begin{pmatrix} -\varkappa \\ 0 \\ k \end{pmatrix} e^{i(kx + \varkappa z - \omega t)} + \frac{B}{\omega}\begin{pmatrix} k \\ 0 \\ -\varkappa \end{pmatrix} \times \begin{pmatrix} \varkappa \\ 0 \\ k \end{pmatrix} e^{i(kx - \varkappa z - \omega t)}$$

$$= \frac{2B \cdot |\vec{k}|}{c} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \cos(\varkappa z) \cdot e^{i(kx - \omega t)}$$

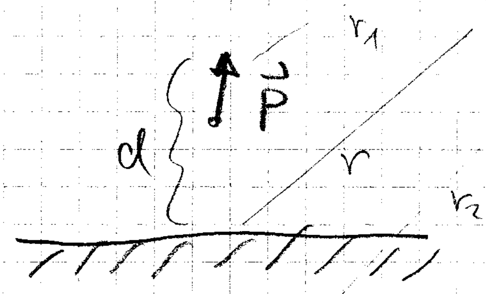
b) Iz simetrije vidimo, da pridejo v poštev samo rešitve z "navpičnim" poljem $\vec{B}$: TE-rešitve ($E_\varphi = 0$)

(i) rešitev $n=1$ je vodilna:

$$\omega_{min.} = \boxed{c \cdot \frac{\pi}{d}}$$

(ii) rešitev $n=2$ $\qquad \omega_{min.} = \boxed{c \cdot \frac{2\pi}{d}}$

2)



Ravnino nadomestimo z zrc. dipolom

$\uparrow \vec{p}$

$$r_1 = r - d \cdot \cos\vartheta$$
$$r_2 = r + d \cdot \cos\vartheta$$

$$\vec{p} = ea \cdot e^{-i\omega t} \cdot \hat{e}_z$$

$$\vec{B} = \frac{\mu_0 \omega^2}{4\pi c \cdot r} \left(\vec{n} \times \ddot{\vec{p}}\left(t - \frac{r_1}{c}\right) + \vec{n} \times \ddot{\vec{p}}\left(t - \frac{r_2}{c}\right) \right)$$

$$= -\frac{\mu_0 \omega^2 ea}{4\pi c \cdot r} \cdot \sin\vartheta \cdot \hat{e}_\varphi \left(e^{i(kr - \omega t - kd \cdot \cos\vartheta)} + e^{i(kr - \omega t + kd \cos\vartheta)} \right)$$

$$= -\frac{\mu_0 \omega^2 ea}{4\pi c \cdot r} \sin\vartheta \cdot \hat{e}_\varphi \cdot 2 \cdot \cos(kd \cdot \cos\vartheta)\, e^{i(kr - \omega t)}$$

$$\vec{E} = c \cdot \left(-\vec{n} \times \vec{B}\right) = -\frac{\mu_0}{4\pi r} \frac{\mu_0 \omega^2 e a}{4\pi c \cdot r} \sin\vartheta \, \hat{e}_\vartheta \cdot$$

$$\cdot 2 \cdot \cos\left(kd \cdot \cos\vartheta\right) e^{i(kr - \omega t)}$$

$$\vec{S} = \frac{1}{2\mu_0} \cdot \vec{E} \times \vec{B}^* = \frac{1}{2\mu_0} \cdot \frac{\mu_0^2 \omega^4 e^2 a^2}{4^2\pi^2 c \, r^2} \sin^2\vartheta \cdot 4 \cdot \cos^2(kd \cos\vartheta) \cdot \hat{n}$$

$$\frac{dP}{d\Omega} = |\vec{S}| \cdot r^2 = \frac{\mu_0 \omega^4 e^2 a^2}{8\pi^2 \cdot c} \sin^2\vartheta \cdot \cos^2(kd \cdot \cos\vartheta)$$

$$P = \int_0^1 d(\cos\vartheta) \cdot 2\pi \cdot \frac{dP}{d\Omega}(\vartheta) =$$

(!) Pol prostor?

$$= \frac{\mu_0 \omega^4 e^2 a^2}{4\pi \cdot c} \cdot \underbrace{\int_0^1 (1 - x^2) \cdot \underbrace{\cos^2(kd \cdot x)}_{\frac{1}{2}(1 + \cos(2kdx))} dx}$$

$$\int_0^1 \left(\frac{1}{2} - \frac{x^2}{2} + \frac{1}{2} \cos(2kdx) - \frac{x^2}{2} \cdot \cos(2kdx)\right) dx$$

$$= \left(\frac{x}{2} - \frac{x^3}{6} + \frac{1}{4kd} \cdot \sin(2 \cdot k \cdot d \cdot x)\right.$$

$$\left. - \frac{2x}{2(2 \cdot k \cdot d)^2} \cdot \cos(2kdx) - \frac{1}{2}\left(\frac{x^2}{(2kd)} - \frac{2}{(2kd)^3}\right) \cdot \sin(2kdx)\right)\Big|^1$$

$$= \frac{1}{3} - \frac{1}{(2kd)^2} \cos(2kd) + \frac{1}{(2kd)^3} \cdot \sin(2kd) =$$

$$P = \frac{\mu_0 \omega^4 e^2 a^2}{4\pi c}\left(\frac{1}{3} - \frac{1}{(2kd)^2}\cdot\cos(2kd) + \frac{1}{(2kd)^3}\cdot\sin(2kd)\right)$$

Povsod velja $\underline{k = \frac{\omega}{c}}$.

3) $$\mathcal{L} = \frac{1}{2} m \vec{v}^2 - e\phi + e\vec{v}\vec{A} = \qquad \vec{A} = \frac{1}{2} r \cdot B \cdot \hat{e}_\varphi$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + \alpha\frac{e}{r^2} + e\cdot r\cdot\dot{\varphi}\cdot\frac{1}{2} r B$$

L neodvisen od $\varphi \Rightarrow$

$p_\varphi =$ konst.

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = m r^2 \dot{\varphi} + \frac{eBr^2}{2} \qquad \Bigg|\; p_\varphi = \frac{eBR^2}{2}$$

$$\mathcal{H} = \dot{r}\cdot\frac{\partial L}{\partial\dot{r}} + \dot{\varphi}\frac{\partial L}{\partial\dot{\varphi}} - L =$$

$$= \dot{r}\cdot m\dot{r} + m r^2\dot{\varphi}^2 + \frac{eBr^2}{2}\dot{\varphi} - \frac{1}{2} m\dot{r}^2 - \frac{1}{2} m r^2 \dot{\varphi}^2 - \alpha\frac{e}{r^2} - e r^2\dot{\varphi} B/2$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) - \alpha\frac{e}{r^2} = \frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{mr^2}{(mr^2)^2}\left(p_\varphi - \frac{eBr^2}{2}\right)^2 - \alpha\frac{e}{r^2}$$

d))

$$\mathcal{H}(r=R) = \mathcal{H}(r=r_0)$$

V obeh ekstremih je

$\underline{\dot{r}=0}$

$$-\frac{\alpha e}{R^2} = -\frac{\alpha e}{r_0^2} + \frac{1}{2}\frac{1}{m r_0^2}\left(\frac{eB}{2}\cdot R^2 - \frac{eB}{2} r_0^2\right)^2$$

$$2m\alpha e \underbrace{\left(\frac{1}{r_0^2} - \frac{1}{R^2}\right)} = \left(\frac{eB}{2}\right)^2 \frac{1}{r_0^2}\left(R^2 - r_0^2\right)^2$$

$$\frac{R^2 - r_0^2}{r_0^2 R^2}$$

$$\frac{2m\alpha e \cdot 4}{e^2 B^2} \frac{\cancel{R^2 - r_0^2}}{\cancel{r_0^2} R^2} = \frac{1}{\cancel{r_0^2}}\left(R^2 - r_0^2\right)^{\cancel{2}}$$

$$\boxed{r_0^2 = R^2 - \frac{8m\alpha}{eB^2R^2}}$$

Rešljivo za

$$B > \sqrt{\frac{8m\alpha}{eR^4}}$$

$$\left(\frac{eBr^2}{2}\right)^2$$