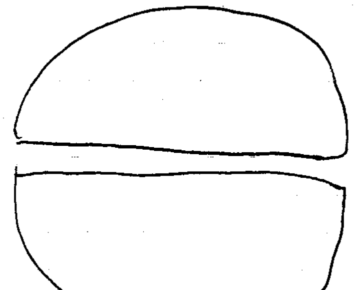
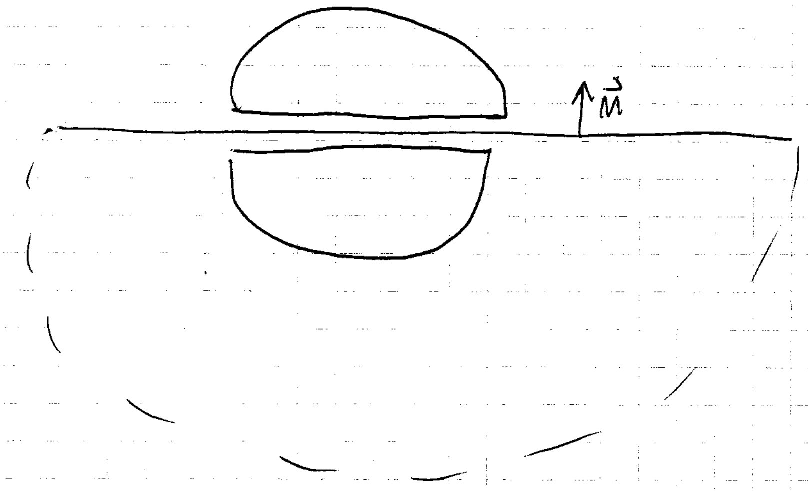
1)



Krogelni polovici sklenjeni skupaj ustvarjata isto polje, kot ena sama krogla:

$$\vec{E}(\vec{r}) = \frac{e \cdot \hat{e}_r}{4\pi\varepsilon_0 \cdot r^2} \qquad \text{za } \underline{r > R}$$

$$\vec{E}(\vec{r}) = 0 \qquad \text{za } r < R$$



Sila na SPODNJO polovico izračunamo kot integral tenzorja T preko ploskve na sliki.

$$\vec{F} = \int \overset{\leftrightarrow}{T} \cdot \vec{n} \, dS$$

oziroma

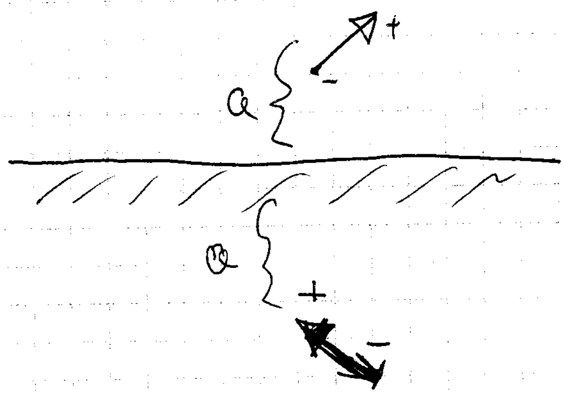
$$F_z = \int T_{zz} \, dS \qquad T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right)$$

V ravnini je $E_z = 0$, zato je $T_{zz} = -\frac{1}{2}\varepsilon_0 \cdot E^2$

$$F_z = \int_R^\infty 2\pi r \left(-\frac{1}{2} \varepsilon_0 E^2(r) \right) dr = \int_R^\infty 2\pi r \cdot \frac{-\varepsilon_0}{2} \frac{e^2}{(4\pi)^2 \varepsilon_0^2 r^4} \, dr$$

$$= \boxed{-\frac{e^2}{32\pi\varepsilon_0 \cdot R^2}} \Rightarrow \text{Sila je } \frac{e^2}{32\pi\varepsilon_0 R^2}$$

2) Prevodno plesker nadomestimo z zrcalno sliko dipola:



$$\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \Rightarrow \vec{p}\,' = \begin{pmatrix} -p_x \\ -p_y \\ p_z \end{pmatrix}$$

Energija dipola v polju svoje zrcalne slike je

$$W = -\frac{1}{2}\,\vec{p}\cdot\vec{E}\,' = -\frac{1}{2}\,\frac{3(\vec{p}\,\hat{e}_r)(\vec{p}\,'\cdot\hat{e}_r) - \vec{p}\,\vec{p}\,'}{4\pi\varepsilon_0 r^3}$$

$\frac{1}{2}$, ker gre za polje, ki ga dipol inducira sam. Glej npr. poglavje 3.21.2 v skripti.

$$= -\frac{1}{2}\,\frac{3p_z p_z' - p_x p_x' - p_y p_y' - p_z p_z'}{4\pi\varepsilon_0 (2a)^3}$$

$$= -\frac{1}{2}\,\frac{2p_z^2 + p_x^2 + p_y^2}{4\pi\varepsilon_0 \cdot (2a)^3}$$

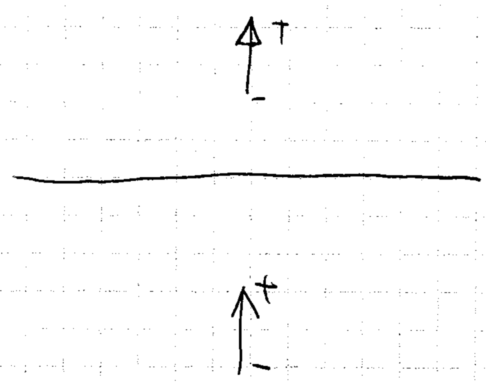
Izraz je minimalen, kadar je $\vec{p} = \begin{pmatrix} 0 \\ 0 \\ p_z \end{pmatrix}$.

Torej $\vec{p} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$ ali $\vec{p} = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix}$

b) $$\Delta W = W\left[\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}\right] - W\left[\begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}\right]$$

$$\Delta W = -\frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0 (2a)^3} \left(p^2 - 2p^2 \right) = \frac{p^2}{64\pi\varepsilon_0 a^3}$$

c) i)



Potencial enega dipola:

$$\varphi = \frac{\vec{p}(\vec{r} - \vec{r}_0)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_0|^3}$$

Potencial dveh:

$$\varphi = \frac{\vec{p}(\vec{r} - a\cdot\hat{e}_z)}{4\pi\varepsilon_0 |\vec{r} - a\hat{e}_z|^3} + \frac{\vec{p}'(\vec{r} + a\cdot\hat{e}_z)}{4\pi\varepsilon_0 |\vec{r} + a\hat{e}_z|^3}$$

$$\vec{p} = \vec{p}' = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix};$$

$$\varphi \approx \frac{2\vec{p}\cdot\hat{e}_z\cdot\vec{r}}{4\pi\varepsilon_0 |\vec{r}|^3} = \boxed{\frac{2\cdot p\cdot z}{4\pi\varepsilon_0 r^3}}$$

Dipolna momenta se seštejeta.

ii) privzamimo $\vec{p} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} = p \cdot \hat{e}_x$

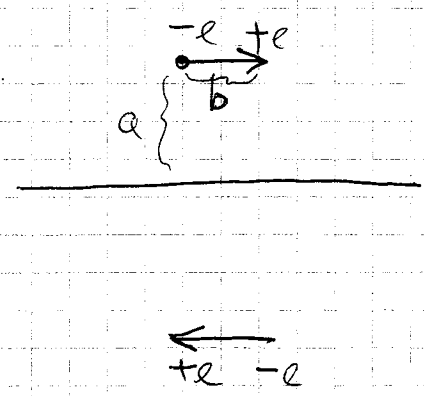
$$\varphi = \frac{p_x \cdot \hat{e}_x \left(\vec{r} - \cancel{a \hat{e}_z}\right)}{4\pi\varepsilon_0 \left|\vec{r} - a \cdot \hat{e}_z\right|^3} + \frac{-p_x \cdot \hat{e}_x \left(\vec{r} + \cancel{a \hat{e}_z}\right)}{4\pi\varepsilon_0 \left|\vec{r} + a \cdot \hat{e}_z\right|^3}$$

$$\frac{1}{\left|\vec{r} - a \cdot \hat{e}_z\right|^3} \approx \frac{1}{r^3} + \frac{3 \cdot a \cdot \vec{r} \cdot \hat{e}_z}{r^5}$$

$$\varphi \approx \frac{6 \cdot p_x (\hat{e}_x \cdot \vec{r}) \cdot \cancel{4}\, a (\vec{r} \cdot \hat{e}_z)}{4\pi\varepsilon_0 r^5} = \boxed{\frac{6 \cdot p_x \cdot a \cdot xz}{4\pi\varepsilon_0 r^5}}$$

Tako vidimo, da je pogoj $\varphi(z=0)=0$ vedno izpolnjen.

Alternativa



Kvadrupolni moment: $Q_{ij} = \int \rho(x)\, 3\left(x_i x_j - \frac{1}{3}\delta_{ij} x^2\right)$

$Q_{xz} = Q_{zx} = 3\left(ab \cdot e + (-a) b (-e)\right) = 6abe = 6 \cdot a \cdot p$

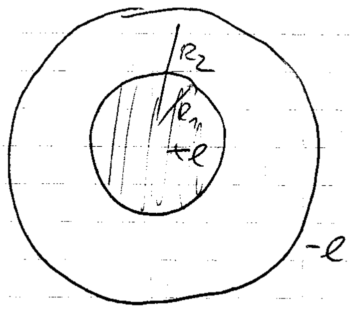
Vsi ostali so 0.

Potencial kvadrupola:

$$\varphi = \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{4\pi\varepsilon_0 |\vec{x}|^5} = \frac{1}{2}Q_{xz}\frac{xz}{4\pi\varepsilon_0 r^5} + \frac{1}{2}Q_{zx}\frac{zx}{4\pi\varepsilon_0 r^5}$$

$$= Q_{xz}\frac{xz}{4\pi\varepsilon_0 \cdot r^5} = \frac{6\cdot a\cdot p\cdot x\cdot z}{4\pi\varepsilon_0 \cdot r^5}$$

3)



Mag. polje je polje dipola z momentom:

$$\vec{m} = \frac{4\pi R_1^3}{3}\cdot\vec{M}$$

$$\vec{B}(r>R_1) = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{e}_r)\hat{e}_r - \vec{m}}{r^3}$$

Električno polje je polje točkastega naboja/krogelnega kondenzatorja:

$$\vec{E} = \frac{e}{4\pi\varepsilon_0}\frac{\hat{e}_r}{r^2}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{e\cdot\mu_0 m}{\mu_0\cdot 4\pi\cdot\varepsilon_0\cdot 4\pi}\;\frac{\hat{e}_r\times\left(\cancel{3(\hat{e}_z\hat{e}_r)\hat{e}_r} - \hat{e}_z\right)}{r^5}$$

$$\vec{g} = \frac{1}{c^2}\vec{S}$$

$$\vec{\gamma} = \vec{r}\times\vec{g} = \frac{e\mu_0\cdot\varepsilon_0\mu_0 m}{\mu_0\cdot(4\pi)^2\varepsilon_0}\;\frac{\overbrace{-\hat{e}_r\times(\hat{e}_r\times\hat{e}_z)}^{-\sin\vartheta\cdot\hat{e}_\vartheta}}{r^4}$$

$$\gamma_z = \frac{e \cdot \mu_0 \cdot m}{(4\pi)^2 \cdot r^4} \cdot \sin^2\vartheta$$

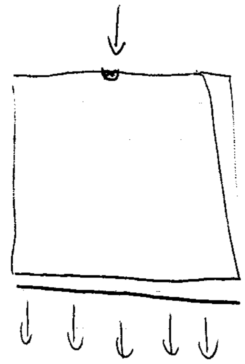
$$\Gamma = \int_{R_1}^{R_2} 2\pi r^2 \, dr \int_{-1}^{1} d(\cos\vartheta) \; \gamma_z = \int_{R_1}^{R_2} \frac{e \mu_0 m \cdot \cancel{}}{8\pi \quad r^2} \underbrace{\int_{-1}^{1} \sin^2\vartheta \, (d(\cos\vartheta))}_{\int_{-1}^{1} (1-u^2)\, du \,=\, u\text{-}}$$

$$= \frac{e \mu_0 m}{8\pi} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \cdot \frac{4}{3}$$

$$= \frac{2 \cancel{4} \, \mu_0 \, M \cdot R_1^3}{9} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\omega = \frac{\cancel{4}\,\Gamma}{J}$$

4)



$\vec{j} = \sigma \cdot \vec{E}$

$\nabla \vec{j} = 0 \Rightarrow \nabla \vec{E} = 0 \Rightarrow \underline{\vec{E} = \nabla \varphi}$

z $\underline{\Delta\varphi = 0}$

i) znotraj kocke velja $\underline{\Delta\varphi = 0}$

ii) Na površini velja $\vec{j} \cdot \vec{n} = 0 \Rightarrow \underline{(\nabla \varphi)\vec{n} = 0}$

iii) Pri zgornji elektrodi velja $\underline{j_z = -I \cdot \delta(x) \cdot \delta(y)}$

i) Potencial rešimo z nastavkom,

$$\varphi = e^{ikx} \cdot e^{ily} \cdot e^{\gamma z}$$

ki izpolnjuje $\Delta\varphi = 0$, če je

$$\underline{-k^2 - l^2 + \gamma^2 = 0}$$

ii) Robni pogoji so $\frac{\partial \varphi}{\partial x} = 0 \Big|_{x = -\frac{a}{2}, \frac{a}{2}}$

$$\frac{\partial \varphi}{\partial y} = 0 \Big|_{y = -\frac{a}{2}, \frac{a}{2}}$$

in so izpolnjeni z $\tilde{\varphi}_{nm} = A_{nm} \cdot \cos\left(n \cdot \frac{2\pi}{a} \cdot x\right) \cos\left(m \cdot \frac{2\pi}{a} \cdot y\right) e^{\pm\gamma z}$

kjer je $\gamma_{nm} = \frac{2\pi}{a} \cdot \sqrt{n^2 + m^2}$.

Tu smo upoštevali samo sode rešitve v x in y, zaradi simetrije $x \Leftrightarrow -x$.

Robni pogoj pri $z=0$ je $\underline{\varphi = 0}$,

zato imajo rešitve obliko:

$$\varphi_{nm} = A_{nm} \cdot \cos\left(n \frac{2\pi x}{a}\right) \cos\left(m \cdot \frac{2\pi y}{a}\right) \cdot \sinh\left(\gamma_{nm} \cdot z\right)$$

Nadalje mora veljati

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=a} = \frac{I}{\sigma} \cdot \delta(x) \cdot \delta(y)$$

$$\left.\varphi'\right|_{z=a} = \sum_{n,m} A_{nm} \cdot \cos\left(n \frac{2\pi x}{a}\right) \cos\left(m \frac{2\pi y}{a}\right) \cdot \gamma_{nm} \cdot \cosh\left(\gamma_{nm} \cdot a\right) = \frac{I}{\sigma} \cdot \delta(x) \delta(y);$$

$$\cancel{\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \cos\left(N \frac{2\pi x}{a}\right) \cos\left(M \frac{2\pi y}{a}\right) \varphi'(x,y,a)\, dx\, dy}$$

$$\cancel{= A_{NM} \cdot \frac{a^2}{4} \cdot \gamma_{NM} \cdot \cosh\left(\gamma_{NM} \cdot a\right) = \frac{I}{\sigma}}$$

$$\cancel{\Rightarrow}$$

$$\cancel{A_{nm} = \frac{I}{\sigma a^2} \cdot \frac{1}{\frac{2\pi}{a}\sqrt{n^2+m^2}} \cdot \frac{1}{\cosh\left(\frac{2\pi}{a}\sqrt{n^2+m^2}\right)}}$$

[illegible]

Za desno stran uporabimo vrsto

$$\delta(x) = \sum_{k=-\infty}^{\infty} e^{i \cdot 2\pi k x} = \sum_{k=-\infty}^{\infty} \cos(2\pi k \cdot x)$$

Oziroma

$$a \cdot \delta(x) = \sum_{k=-\infty}^{\infty} \cos(2\pi k x/a)$$

Robni pogoj je izpolnjen, če velja

$$A_{kl} = \frac{I}{\sigma a^2} \frac{1}{\frac{2\pi}{a}\sqrt{k^2+l^2}} \cdot \frac{1}{\cosh\left(2\pi\sqrt{n^2+m^2}\right)}$$

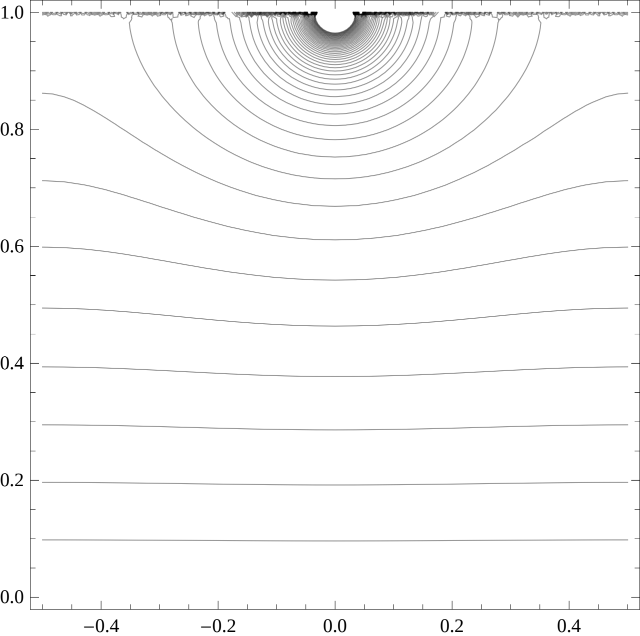
N.B. členi so podvojeni, vendar je tako notacija enostavnejša.

A_{00} moramo obravnavati ločeno:

$\varphi_{00} = A_{00} \cdot z$ je ustrezna rešitev, in r.p. je izpolnjen z

$$A_{00} = \frac{I}{\sigma a^2}$$

$$\boxed{\varphi(x,y,z) = \sum_{k,l \neq (0,0)} A_{kl} \cdot \cos\left(k \cdot \frac{2\pi x}{a}\right) \cos\left(l \cdot \frac{2\pi y}{a}\right) \sinh\left(\sqrt{k^2+l^2}\,\frac{2\pi z}{a}\right) + \frac{I}{\sigma a^2} \cdot z}$$


1.0
0.8
0.6
0.4
0.2
0.0
−0.4
−0.2
0.0
0.2
0.4